SECTION 7.6: L'HOPITAL'S RULE

RECALL: If a function is differentiable at $x = a$, then f is locally linear near $x = a$. Geometrically, this means the graph of can be closely approximated by the graph of the tangent line at $(a, f(a))$. That is when x is 'near' a:

$$f(x) \approx f'(a)(x-a) + f(a)$$

EXAMPLE 1 (VIDEO): Let $f(x) = \sin(x)$ and $g(x) = e^{2x} - 1$.

1. Find the equations to the tangent lines to the graphs of f and g at $x = 0$.

   Tangent Line to $y = f(x)$ at $x = 0$: Ans: $y = f'(0)(x-0) + f(0) = (1)(x-0) + 0 = x$ so $y = x$.

   Tangent Line to $y = g(x)$ at $x = 0$: Ans: $y = g'(0)(x-0) + g(0) = (2)(x-0) + 0 = 2x$ so $y = 2x$.

2. Consider: $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{\sin(x)}{e^{2x}-1}$. Show this limit results in the indeterminate form $\dfrac{0}{0}$.

3. Evaluate this limit by replacing $f(x)$ and $g(x)$ by the equations of their tangent lines at $x = 0$.

   Near $x = 0$, $f(x) \approx x$ and $g(x) \approx 2x$ so $\dfrac{f(x)}{g(x)} \approx \dfrac{x}{2x} = \dfrac{1}{2}$. Hence, $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0} \dfrac{\sin(x)}{e^{2x}-1} = \dfrac{1}{2}$.

Now suppose, in general, f and g are differentiable at $x = a$ and that $f(a) = g(a) = 0$, but that neither $g(x)$ nor $g'(x)$ is 0 on some open interval containing $x = a$ (except at $x = a$.) Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\text{tangent line to } y = f(x) \text{ at } x = a}{\text{tangent line to } y = g(x) \text{ at } x = a} = \lim_{x\to a} \frac{f'(a)(x-a) + f(a)}{g'(a)(x-a) + g(a)}$$

Since $f(a) = g(a) = 0$, $\dfrac{f'(a)(x-a) + f(a)}{g'(a)(x-a) + g(a)} = \dfrac{f'(a)(x-a)}{g'(a)(x-a)} = \dfrac{f'(a)}{g'(a)}$, so it stands to reason: $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

This is a very loose derivation of a rule known as **L'Hopital's Rule**.

More generally, if a limit $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ results in the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

EXAMPLE 2: Use L'Hopital's Rule to evaluate the following limits:

1. $\lim\limits_{x\to 1} \dfrac{\cos(\pi x) + 1}{\sqrt{x} - 1}$ Ans: 0

2. $\lim\limits_{x\to 1} \dfrac{2\ln(x)}{x^2 - 1}$ Ans: 1

3. $\lim\limits_{t\to 0} \left(e^t - t - 1\right) t^{-2}$ Ans: $\dfrac{1}{2}$

L'Hopital's Rule also extends to limits at infinity.

EXAMPLE 3: Use L'Hopital's Rule to evaluate the following limits.

1. $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{x}$ Ans: 0

2. $\lim\limits_{x\to\infty} x^2 e^{-x}$ Ans: 0

EXAMPLE 4: (VIDEO) What goes wrong when you attempt to use L'Hopital's Rule to determine: $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x^2-1}}{2x}$?

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}}{2x} = \lim_{x\to\infty} \frac{\frac{1}{2}(3x^2-1)^{-1/2}(6x)}{2} = \lim_{x\to\infty} \frac{3x}{2(3x^2-1)^{1/2}} = \lim_{x\to\infty} \frac{3}{2\left(\frac{1}{2}\right)(3x^2-1)^{-1/2}(6x)} = \ldots = \lim_{x\to\infty} \frac{\sqrt{3x^2-1}}{2x}$$

The limit cycles back to itself after two applications of L'Hopital's Rule!

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x^2-1}}{2x}$ using algebraic means.

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}}{2x} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(3-\frac{1}{x^2}\right)}}{2x} = \lim_{x\to\infty} \frac{|x|\sqrt{3-\frac{1}{x^2}}}{2x} = \lim_{x\to\infty} \frac{x\sqrt{3-\frac{1}{x^2}}}{2x} == \lim_{x\to\infty} \frac{\sqrt{3-\frac{1}{x^2}}}{2} = \frac{\sqrt{3}}{2}.$$

INDETERMINATE FORMS INVOLVING POWERS

Consider the limit: $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$.

This limit has the form 1^∞. If the base approaches 1 faster than the exponent approaches ∞, this limit will approach 1. If, on the other hand, the exponent approaches ∞ faster than the base approaches 1, the limit will approach ∞, since the base is always greater than 1. If the base approaches 1 at more or less the same rate as the exponent approaches ∞, then the limit will land somewhere between 1 and ∞.

For these reasons, the form 1^∞ is another indeterminate form.

We let $y = \left(1+\dfrac{1}{x}\right)^x$ and investigate $\ln(y) = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right)$.

Note $\lim\limits_{x\to\infty} x\ln\left(1+\dfrac{1}{x}\right)$ is of the indeterminate form $\infty\cdot 0$, so we rewrite as $\lim\limits_{x\to\infty} \dfrac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}}$ which is of the form $\dfrac{0}{0}$.

Using L'Hopital's Rule, we find:

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} = \lim_{x\to\infty} \frac{\dfrac{1}{\left(1+\frac{1}{x}\right)}\left(-x^{-2}\right)}{-x^{-2}} = \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)} = 1$$

Since $\lim\limits_{x\to\infty} x\ln\left(1+\dfrac{1}{x}\right) = \lim\limits_{x\to\infty} \ln(y) = 1$, we can use properties of logarithms to find $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$:

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = e^{\lim\limits_{x\to\infty} \ln(y)} = e^1 = e$$

Indeterminate forms involving exponents such as 1^∞, 0^0, or ∞^0 are best analyzed by first taking the natural log to convert it to an indeterminate form involving multiplication, and then rewriting to use L'Hopital's Rule.

EXAMPLE 5 (VIDEO): Determine the following limit: $\lim\limits_{x\to 0^+} x^{\frac{\ln(117)}{x+\ln(x)}}$ Ans: 117

RELATIVE GROWTH RATES OF FUNCTIONS

Any time you encounter an indeterminate form, you can think of the two functions involved as being in a race to see which function arrives at their destination first.

For example, in the limit $\lim\limits_{x\to\infty} xe^{-2x}$, $f(x) = x \to \infty$ and $g(x) = e^{-2x} \to 0$.

This produces the indeterminate form $\infty \cdot 0$. Going through the calculations, we find:

$$\lim_{x\to\infty} xe^{-2x} = \lim_{x\to\infty} \frac{x}{e^{2x}} = \lim_{x\to\infty} \frac{1}{2e^{2x}} = 0$$

In the race between $f(x) = x$ and $g(x) = e^{-2x}$ in $\lim\limits_{x\to\infty} xe^{-2x}$, $g(x) = e^{-2x}$ wins since $\lim\limits_{x\to\infty} xe^{-2x} = \lim\limits_{x\to\infty} g(x) = 0$.

EXAMPLE 6: (VIDEO) For each of the limits below, think of the limit as a contest between two functions.

Where are the functions going? Which function 'wins'? Are there any 'ties'?

1. $\lim\limits_{x\to\infty} \dfrac{2x-1}{1-x^2}$

   Ans: $g(x) = 1 - x^2$ grows infinite faster than $f(x) = 2x - 1$ which is why $\dfrac{f(x)}{g(x)} = \dfrac{2x-1}{1-x^2} \to 0$.

2. $\lim\limits_{x\to\infty} \dfrac{2x^2-1}{1-x^2}$

   Ans: $f(x) = 2x^2 - 1$ and $g(x) = 1 - x^2$ grow infinite at similar rates so $\dfrac{f(x)}{g(x)} = \dfrac{2x^2-1}{1-x^2} \to -2$, a 'tie.'

3. $\lim\limits_{x\to 0^+} x\ln(x)$

   Ans: $f(x) = x$ approaches 0 faster than $g(x) = \ln(x)$ approaches $-\infty$, so $f(x)g(x) = x\ln(x) \to 0$.

4. $\lim\limits_{x\to 0} \dfrac{x}{\tan(4x)}$

   Ans: $f(x) = x$ and $g(x) = \tan(4x)$ both approach 0 at similar rates so $\dfrac{f(x)}{g(x)} = \dfrac{x}{\tan(4x)} \to \dfrac{1}{4}$, a 'tie'.

A more rigorous proof of L'Hopital's Rule requires the **EXTENDED MEAN VALUE THEOREM**. We discuss the EMVT on a video. Geometrically, its conclusion is best left until we study parametric equations later ...

HOMEWORK: Section 7.6: 7 - 51 odd, 61*